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FU'S p^α DOTS BRACELET PARTITION

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Abstract. This paper aims to explore the arithmetic properties of Fu's k dots bracelet partition where $k = p^\alpha$, p is a prime number with $p \geq 5$ and α is an integer with $\alpha \geq 0$. For p^α dots bracelet partitions with $p = 5, 7$ and 11, we found several exciting Ramanujan-like congruences modulo p. We also used Newman's theorems to demonstrate certain congruence modulo p.

1. Introduction

In his acclaimed work Combinatory Analysis [22], P. A. MacMahon pioneered partition analysis as a computational approach for tackling combinatorial questions affecting systems of linear diophantine inequalities and equations.

Andrews et al. [2,3,5–13] studied partition functions through MacMohan's partition analysis. To define the k dots bracelet partition, we have to start with the plane partition, treated by MacMahon in [22]; this is the scenario in which the partition's non-negative integer components c_i are positioned at the corners of a square in such a way that the following order relations hold:

$$c_1 \geq c_2, \quad c_1 \geq c_3, \quad c_2 \geq c_4, \quad \textit{and} \quad c_3 \geq c_4. \tag{1.1}$$

It is assumed here and throughout this paper that the arrow leading from c_i to c_j is represented as $c_i \geq c_j$, the graphical representation of relations (1.1) shown in Figure 1.

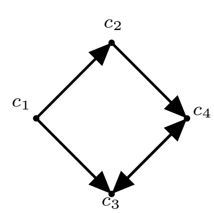

Figure 1. Graphical representation of (1.1).

In 2007, Andrews and Paul [13] proposed a generalization of the diamond shape called the k-elongated partition diamonds as shown in Figure 2. Then they defined

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the broken k-diamond partition, consisting of two separated k-elongated partition diamonds of length n where the source is deleted in one of them, as shown in Figure 3.

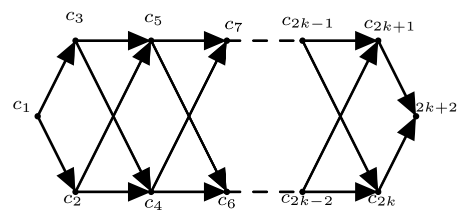

FIGURE 2. k-elongated partition diamond of length 1.

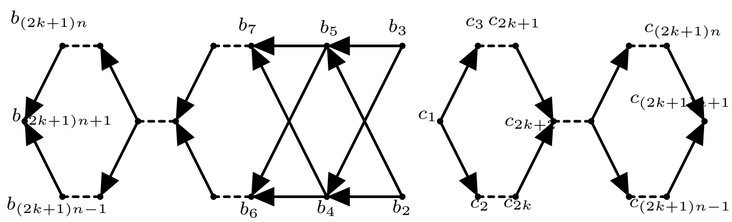

FIGURE 3. Broken k-diamond of length $2n$.

Andrews and Paul [13] found the generating function for the broken k-diamond partition, let $\Delta_k(n)$ be the total number of broken k-diamond partitions for any positive integer n, then for $n \geq 0$ and $k \geq 1$,

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{f_2\, f_{2k+1}}{f_1^3\, f_{4k+2}},$$

where $f_k = (q^k; q^k)_\infty = \prod_{n=0}^{\infty} (1 - q^{kn}), \quad |q| < 1.$

In 2011, Fu [19] generalized the broken k-diamond partition, which he called the k dots bracelet partition. He initially defined infinite bracelet partitions rather than k dots bracelet partitions. Figure 4 displays bracelet partitions made of repeating diamonds and dots, with $k-2$ dots between two successive diamonds. And we see that an infinite bracelet partition can be cut into $k-1$ different ways with k dots in half. For any $k \geq 3$, a k-dots bracelet partition consists of $k-1$ different half bracelets as shown in Figure 5.

Let $\mathfrak{B}_k(n)$ denote the number of k dots bracelet partition for positive integer n, the generating function for $\mathfrak{B}_k(n)$, $k \geq 3$ is given by (See [19]):

$$\sum_{n=0}^{\infty} \mathfrak{B}_k(n) q^n = \frac{f_2\, f_k}{f_1^k\, f_{2k}}. \tag{1.2}$$

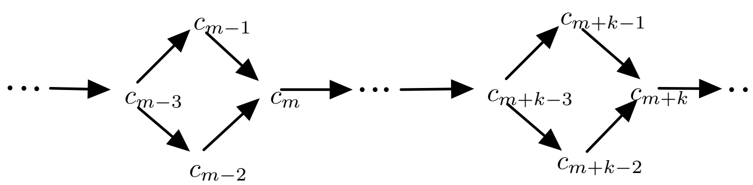

FIGURE 4. Infinite bracelet with k dots.

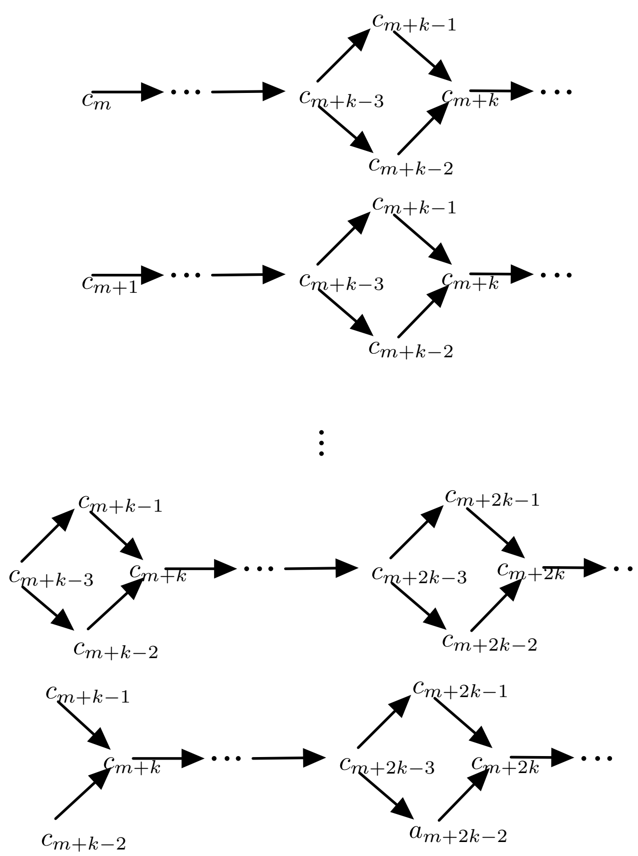

FIGURE 5. $k-1$ different half bracelet.

He also proved the following congruences for k dots bracelet partitions:

(i) For $n \geq 0$, If $k = p^r \geq 3$ is a prime power,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{p}.$$

(ii) For $n \geq 0$, $k \geq 3$, and $1 \leq s \leq p-1$ such that $12s+1$ is a quadratic nonresidue modulo p, if $p \mid k$ for some prime $p \geq 5$,

$$\mathfrak{B}_k(pn+s) \equiv 0 \pmod{p}.$$

(iii) For $n \geq 0$ and $k \geq 3$ even, say $k = 2^m l$, where l is odd,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{2^m}.$$

Radu and Sellers [25] found Some new Ramanujan like congruence for $\mathfrak{B}_k(n)$,

$$\mathfrak{B}_5(10n+7) \equiv 0 \pmod{5^2},$$

$$\mathfrak{B}_7(14n+11) \equiv 0 \pmod{7^2},$$

$$\mathfrak{B}_{11}(22n+21) \equiv 0 \pmod{11^2}.$$

Later, Cui and Gu [18] found some congruence modulo 2 for 5 dots bracelet partition and modulo $p \geq 5$ for k dots bracelet partitions, Xia and Yao [27] also found several congruences modulo 2 for 5 dots bracelet partition, and Baruah and Ahmed [14] found congruence modulo p^2 and p^3 for k dots bracelet partitions with $k = mp^s$ for $s \geq 2$ and $s \geq 3$, respectively.

A *partition* of a positive integer n is a finite non-increasing sequence of positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{f_1}.$$

The most inspiring congruences of $p(n)$ discovered by Ramanujan [26] for $n \geq 0$ are:

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.3}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.4}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.5}$$

Let m be a positive integer with $m \geq 1$. A partition of n is called an m-regular partition, if none of its part is divisible by m. If $b_m(n)$ denote the number of m-regular partition of n, the generating function for $b_m(n)$ is given by

$$\sum_{n=0}^{\infty} b_m(n)q^n = \frac{f_m}{f_1}. \tag{1.6}$$

Several mathematicians study the arithmetic properties of m-regular partition (for example, [4, 15, 16, 20, 21]).

In this paper, we extend our investigation of the arithmetic properties of the k dots bracelet partitions, where $k = p^\alpha$ for all $\alpha \geq 0$ and $p \geq 5$ is a prime number. Our fundamental goal in this paper is to show the following theorems, thus expanding the family of congruences modulo p for $p \geq 5$, the authors mentioned above for k dots bracelet partitions. The main theorems of this paper are the followings:

Theorem 1.1. *Let p be a prime with $p \geq 5$. Then for $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} \mathfrak{B}_{p^{2\alpha+1}}\left(2 \cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{12}\right) q^n \equiv (-1)^{\frac{(\alpha+1)(\pm p-1)}{6})} f_1^{p-1} \pmod{p}, \quad (1.7)$$

$$\sum_{n=0}^{\infty} \mathfrak{B}_{p^{2\alpha}}\left(p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{12}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} \pmod{p}, \quad (1.8)$$

$$\text{where } \frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6} \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$$

Corollary 1.2. *Let p be a prime with $p \geq 5$. Then for $\alpha \geq 0$,*

$$\mathfrak{B}_{p^{2\alpha+1}}\left(2 \cdot p^{2\alpha+1}n + p^{2\alpha+1} + \frac{p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{p}. \quad (1.9)$$

Corollary 1.3. *Let p be a prime with $p \geq 5$. Then for $\alpha \geq 0$,*

$$\mathfrak{B}_{p^{2\alpha+1}}\left(2 \cdot p^{3\alpha+1}n + \frac{2 \cdot p^{2\alpha+2} + p^{\alpha+1} - p^{\alpha} - p - 1}{12}\right) \equiv \quad (1.10)$$

$$(-1)^{\frac{(\alpha+1)(\pm p-1)}{6})} b_p(n) \pmod{p},$$

where $\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod{6} \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod{6}. \end{cases}$ *and $b_p(n)$ is p-regular partition.*

Here and throughout assume that, $\sum_{n=0}^{\infty} P_r(n) q^n = f_1^r$, $r \geq 1$.

Theorem 1.4. *Let $r = p-1$, where p be a prime with $5 \leq p \leq 23$. If $P_r\left(\frac{(p-1)^2}{24}\right) \equiv 0 \pmod{p}$, then for all $\alpha \geq 0$,*

$$\mathfrak{B}_{p^{2\alpha+1}}\left(2 \cdot p^{2\alpha+2}n + \frac{p^{2\alpha+1}(p-1)^2 + p^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{p}. \quad (1.11)$$

Theorem 1.5. *For $\alpha \geq 0$,*

$$\mathfrak{B}_{7^{2\alpha+1}}\left(2 \cdot 7^{2\alpha+2}n + 24 \cdot 7^{2\alpha+1} + \frac{5^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{7}, \quad (1.12)$$

$$\mathfrak{B}_{11^{2\alpha+1}}\left(2 \cdot 11^{2\alpha+2}n + 100 \cdot 11^{2\alpha+1} + \frac{11^{2\alpha+2}-1}{12}\right) \equiv 0 \pmod{11}. \quad (1.13)$$

This paper is set up as follows. The initial premises necessary to establish our theorems and corollaries are presented in section 2. We prove our main theorems in section 3.

2. Preliminaries

We constructed a few lemmas in this section that are necessary to support our main theorems.

The Jacobi's triple product identity [1, Entry 19] in Ramanujan's notation is given by

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty,\ |ab|<1. \tag{2.1}$$

Lemma 2.1. *Let p prime with $p\geq 5$,*

$$f_2=\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{3k^2+k}f\left(-q^{3p^2+(6k+1)p},-q^{3p^2-(6k+1)p}\right)+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{12}}f_{2p^2}, \tag{2.2}$$

$$\textit{where}\ \frac{\pm p-1}{6}:=\begin{cases}\frac{p-1}{6}, & \textit{if } p\equiv 1 \pmod 6\\ \frac{-p-1}{6}, & \textit{if } p\equiv -1 \pmod 6).\end{cases}$$

Furthermore, if $\frac{-(p-1)}{2}\leq k\leq\frac{p-1}{2}$ and $k\neq\pm\frac{p-1}{6}$, then $3k^2+k\not\equiv\frac{p^2-1}{12} \pmod p$.

Proof. Changing q by q^2 in [17, Theorem 2.2], we obtain the Lemma. ☐

Lemma 2.2. *Let p prime with $p\geq 5$,*

$$\sum_{n=0}^{\infty}\mathfrak{B}_p\left(2\cdot pn+\frac{p^2-1}{12}\right)q^n\equiv(-1)^{\frac{\pm p-1}{6}}f_1^{p-1} \pmod p, \tag{2.3}$$

$$\sum_{n=0}^{\infty}\mathfrak{B}_{p^2}\left(pn+\frac{p^2-1}{12}\right)q^n\equiv(-1)^{\frac{\pm p-1}{6}} \pmod p, \tag{2.4}$$

$$\textit{where}\ \frac{\pm p-1}{6}:=\begin{cases}\frac{p-1}{6}, & \textit{if } p\equiv 1 \pmod 6\\ \frac{-p-1}{6}, & \textit{if } p\equiv -1 \pmod 6).\end{cases}$$

Proof. Note that for any prime number p and any positive integer a, we have

$$f_{ap}\equiv f_a^p \pmod p. \tag{2.5}$$

Now employing (2.5) in (1.2), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_p(n)q^n\equiv\frac{f_2}{f_{2p}} \pmod p. \tag{2.6}$$

Employing Lemma 2.1 in (2.6), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_p(n)q^n\equiv\frac{1}{f_{2p}}\Bigg[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{3k^2+k}f\left(-q^{3p^2+(6k+1)p},-q^{3p^2-(6k+1)p}\right)$$
$$+(-1)^{\frac{\pm p-1}{6}}q^{\frac{p^2-1}{12}}f_{2p^2}\Bigg] \pmod p. \tag{2.7}$$

Consider the congruence

$$3k^2+k\equiv\frac{p^2-1}{12} \pmod p, \tag{2.8}$$

which is equivalent to

$$(6k+1)^2 \equiv 0 \pmod{p}.$$

The congruence (2.8) has a unique solution $k = \dfrac{\pm p - 1}{6}$. So extracting the terms involving $q^{pn+\frac{p^2-1}{12}}$ from (2.7) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_p\left(pn + \frac{p^2-1}{12}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} f_2^{p-1} \pmod{p}. \tag{2.9}$$

Again extarcting the terms involving q^{2n} and repalcing q^2 by q, we obtain (2.3). Similarly by putting $k = p^2$ in (1.2), employing (2.5) and Lemma 2.1, we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_{p^2}\left(pn + \frac{p^2-1}{12}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} \frac{f_{2p}}{f_{2p}} \equiv (-1)^{\frac{\pm p-1}{6}} \pmod{p}. \tag{2.10}$$

□

Lemma 2.3. *[23], Suppose r is even with $0 \le r \le 24$, let p be a prime such that $r(p-1) \equiv 0 \pmod{24}$. Set $\delta = \frac{r(p-1)}{24}$. Then*

$$P_r(np+\delta) = P_r(\delta)P_r(n) - p^{\frac{r}{2}-1} P_r\left(\frac{n-\delta}{p}\right). \tag{2.11}$$

Lemma 2.4. *[24], suppose $r \in \{2, 4, 6, 8, 10, 14, 26\}$. Let p be a prime with $p > 3$, such that $r(p+1) \equiv 0 \pmod{24}$). Set $\Delta = \frac{r(p^2-1)}{24}$ and define $P_r(n)$ as zero if α is not non-negative integer. Then*

$$P_r(np+\Delta) = (-p)^{\frac{r}{2}-1} P_r\left(\frac{n}{p}\right). \tag{2.12}$$

3. Proofs

In this section, we prove Theorem 1.1-1.5, the Ramanujan-like congruences, and the remaining Corollaries.

Proof of Theorem 1.1. (2.3) is the $\alpha = 0$ case of (1.7). Now assume $\alpha \ge 0$. Replacing k by $p^{2\alpha+3}$ in (1.2), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_{p^{2\alpha+3}}(n) q^n = \frac{f_2\, f_{p^{2\alpha+3}}}{f_1^{p^{2\alpha+3}}\, f_{2p^{2\alpha+3}}}. \tag{3.1}$$

Now employing (2.5) and Lemma 2.1 in (3.1), we obtain

$$\sum_{n=0}^{\infty} \mathfrak{B}_{p^{2\alpha+3}}\left(pn + \frac{p^2-1}{12}\right) q^n \equiv (-1)^{\frac{\pm p-1}{6}} \frac{f_{2p}\, f_{p^{2\alpha+2}}}{f_p^{p^{2\alpha+1}}\, f_{2p^{2\alpha+2}}} \pmod{p}. \tag{3.2}$$

Extracting the involving q^{pn} from (3.2) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_{p^{2\alpha+3}}\left(p^2n+\frac{p^2-1}{12}\right)q^n \equiv (-1)^{\frac{\pm p-1}{6}}\frac{f_2\, f_{p^{2\alpha+1}}}{f_1^{p^{2\alpha+1}}\, f_{2p^{2\alpha+1}}} \pmod p$$
$$= (-1)^{\frac{\pm p-1}{6}}\sum_{n=0}^{\infty}\mathfrak{B}_{p^{2\alpha+1}}(n)q^n \pmod p. \quad (3.3)$$

By (1.7), we deduce that

$$\sum_{n=0}^{\infty}\mathfrak{B}_{p^{2\alpha+3}}\left(p^2\left(2\cdot p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{12}\right)+\frac{p^2-1}{12}\right)q^n$$
$$\equiv (-1)^{\frac{(\alpha+2)(\pm p-1)}{6}}\sum_{n=0}^{\infty}\mathfrak{B}_{p^{2\alpha+1}}\left(2\cdot p^{2\alpha+3}n+\frac{p^{2\alpha+4}-1}{12}\right)q^n \pmod p$$
$$\equiv (-1)^{\frac{(\alpha+2)(\pm p-1)}{6}} f_1^{p-1} \pmod p. \quad (3.4)$$

That is, (1.7) is hold for $\alpha+1$. This complete the proof of (1.7).
Since (2.4) is the $\alpha=0$ case of (1.8), we can prove (2.4) by similarly using the mathematical induction as in (2.3). □

Proof of Corollary 1.2. Since there is no terms involving q^{2n+1} in (2.9), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{B}_{p}\left(2\cdot pn+p+\frac{p^2-1}{12}\right)\equiv 0 \pmod p. \quad (3.5)$$

(3.5) is the $\alpha=0$ case of (1.9). By mathematical induction, we can easily prove (1.9). □

Proof of Corollary 1.3. From (1.6) and (2.5),

$$\sum_{n=0}^{\infty} b_p(n)=\frac{f_p}{f_1}\equiv f_1^{p-1} \pmod p. \quad (3.6)$$

Employing (3.6) in (1.7), we obtain (1.10). □

Proof of Theorem 1.4. Set $r=p-1$ in Lemma 2.3, where p be a prime with $5\le p\le 23$, we obtain

$$P_r\left(np+\frac{(p-1)^2}{24}\right)=P_r\left(\frac{(p-1)^2}{24}\right)P_r(n)-p^{\frac{r}{2}-1}P_r\left(\frac{n-\frac{(p-1)^2}{24}}{p}\right).$$
$$\equiv P_r\left(\frac{(p-1)^2}{24}\right)P_r(n) \pmod p. \quad (3.7)$$

$$\text{If}\quad P_r\left(\frac{(p-1)^2}{24}\right)\equiv 0 \pmod p,\ \text{then}\ P_r\left(np+\frac{(p-1)^2}{24}\right)\equiv 0 \pmod p. \quad (3.8)$$

Employing (3.8) in (1.7), we complete the proof of (1.11). □

Proof of Theorem 1.5. Set $r = p-1$, for $p = 7$ and 11. For $p = 7$ and 11, we have $r(p+1) = p^2 - 1 \equiv 0 \pmod{24}$. So by Lemma 2.4 , we obtain

$$P_6(7n+12) \equiv 0 \pmod{7}, \tag{3.9}$$

$$P_{10}(11n+50) \equiv 0 \pmod{11}. \tag{3.10}$$

Employing (3.9) and (3.10) in (1.10), we obtain (1.12) and (1.13), respectively. □

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References

1. Adiga, C., Berndt, B. C., Bhargava, S. and Watson, G. N.: Chapter 16 of Ramanujan's second notebook: theta-functions and q-series, *Mem. Amer. Math. Soc.*, **53**(315) (1985), v+85 pp.
2. Andrews, G. E.: MacMahon's partition analysis. I. The lecture hall partition theorem, in Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), *Progr. Math.*, **161** (1988), 1–22.
3. Andrews, G. E.: MacMahon's partition analysis. II. Fundamental theorems, *Ann. Comb.*, **4**(3 & 4) (2000), 327—338.
4. Andrews, G. E., Hirschhorn, M. D. and Sellers, J. A.: Arithmetic properties of partitions with even parts distinct, *Ramanujan J.*, **23**(1-3) (2010), 169–181.
5. Andrews, G. E. and Paule, P.: MacMahon's partition analysis. IV. Hypergeometric multisums, *Sém. Lothar. Combin.*, **42** (1999), Art. B42i, 24 pages.
6. Andrews, G. E., Paule, P. and Riese, A.: MacMahon's Partition Analysis: The Omega Package, *European J. Combin.*, **22**(7) (2001), 887–904.
7. Andrews, G. E., Paule, P. and Riese, A.: MacMahon's partition analysis. VI. A new reduction algorithm, *Ann. Comb.*, **5**(3 & 4) (2001), 251–270.
8. Andrews, G. E., Paule, P. and Riese, A.: MacMahon's partition analysis. VII. Constrained compositions, in q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), *Contemp. Math.*, **291** (2001), 11–27.
9. Andrews, G. E., Paule, P. and Riese, A.:, MacMahon's partition analysis. VIII. Plane partition diamonds, *Adv. in Appl. Math.*, **27**(2 & 3) (2001), 231–242.
10. Andrews, G. E., Paule, P. and Riese, A.:, MacMahon's partition analysis. IX. k-gon partitions, *Bull. Austral. Math. Soc.*, **64**(2) (2001), 321–329.
11. Andrews, G. E., Paule, P. and Riese, A.:, MacMahon's partition analysis. X. Plane partitions with diagonals, *South East Asian J. Math. Math. Sci.*, **3**(1) (2004), 3–14.
12. Andrews, G. E., Paule, P., Riese, A. and Strehl, V.: *MacMahon's partition analysis. V. Bijections, recursions, and magic squares, in Algebraic combinatorics and applications (Gößweinstein, 1999)*, Springer, Berlin, (2001), 1–39.
13. Andrews, G. E. and Paule, P.: MacMahon's partition analysis. XI. Broken diamonds and modular forms, *Acta Arith.*, **126**(3) (2007), 281–294.
14. Baruah, N. D. and Ahmed, Z.: Congruences modulo p^2 and p^3 for k dots bracelet partitions with $k = mp^s$, *J. Number Theory*, **151** (2015), 129–146.
15. Calkin, N., Drake, N., James, K., Law, S., Lee, P., Penniston, D. and Radder, J.: Divisibility properties of the 5-regular and 13-regular partition functions, *Integers*, **8** (2008), A60, 10 pages.
16. Chen, S. C.: On the number of partitions with distinct even parts, *Discrete Math.*, **311** (2011), 940–943.
17. Cui, S. P. and Gu, N. S. S.: Arithmetic properties of ℓ-regular partitions, *Adv. in Appl. Math.*, **51**(4) (2013), 507–523.
18. Cui, S. P. and Gu, N. S. S.: Congruences for k dots bracelet partition functions, *Int. J. Number Theory*, **9**(8) (2013), 1885–1894.

19. Fu, S.: Combinatorial proof of one congruence for the broken 1-diamond partition and a generalization, *Int. J. Number Theory*, **7**(1) (2011), 133–144.
20. Furcy, D. and Penniston, D.: Congruences for ℓ-regular partition functions modulo 3, *Ramanujan J.*, **27**(1) (2012), 101–108.
21. Hirschhorn, M. D. and Sellers, J. A.: Elementary proofs of parity results for 5-regular partitions, *Bull. Aust. Math. Soc.*, **81**(1) (2010), 58–63.
22. MacMahon, P. A.: *Combinatory analysis, Two volumes (bound as one)*, Chelsea Publishing Co., New York, 1960.
23. Newman, M.: The coefficients of certain infinite products, *Proc. Amer. Math. Soc.*, **4** (1953), 435–439.
24. Newman, M.: An identity for the coefficients of certain modular forms, *J. London Math. Soc.*, **30** (1955), 488–493.
25. Radu, C. S. and Sellers, J. A.: Congruences modulo squares of primes for Fu's k dots bracelet partitions, *Int. J. Number Theory*, **9**(4) (2013), 939–943.
26. Ramanujan, S.: Some properties of $p(n)$, the number of partitions of n, *Proc. Cambridge Philos. Soc.*, **19** (1919), 207–210.
27. Xia, E. X. W. and Yao, O. X. M.: Congruences modulo powers of 2 for Fu's 5 dots bracelet partitions, *Bull. Aust. Math. Soc.*, **89**(3) (2014), 360-372.

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